Vectorial Finite-Element Method Without Any Spurious Solutions for Dielectric Waveguiding Problems Using Transverse Magnetic-Field Component

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Abstract —**An improved finite-element method for the analysis of dielectric waveguiding problems is formulated using the transverse magnetic-field component. In this approach, the divergence relation $\nabla \cdot H = 0$ is satisfied and the spurious, nonphysical solutions which have been necessarily included in the solutions of earlier vectorial finite-element methods are completely eliminated in the whole region of a propagation diagram. To verify the accuracy of the present method, numerical results for a rectangular metallic waveguide half filled with dielectric are presented and compared with exact and earlier finite-element solutions. Dielectric rectangular waveguides are also analyzed for both isotropic and anisotropic cases.**

I. Introduction

Several methods for the analysis of three-dimensional dielectric waveguides have been proposed, and the vectorial finite-element method in an axial-component (E_z–H_z) formulation [1]–[7] or in a three-component (magnetic field H or electric field E) formulation [8]–[10], which enables one to compute accurately the mode spectrum of a waveguide with an arbitrary cross section, is widely used. The most serious difficulty in applying the finite-element method to three-dimensional inhomogeneous dielectric waveguides is the appearance of spurious, nonphysical solutions [1]–[10]. To overcome this difficulty, approaches have recently been developed using all three components of the magnetic or electric field [11]–[17]. Among them, the penalty function method [11]–[13], [15]–[17] has been extensively studied and applied to various types of dielectric waveguides [18]–[23] in which the divergence-free constraint $\nabla \cdot H = 0$ or $\nabla \cdot D = 0$ is satisfied in the least-square sense and the spurious solutions can be suppressed from the guided- or slow-wave region [15]–[17]. However, in this approach an arbitrary positive constant, called the penalty coefficient, is included, and the accuracy of solutions depends on its magnitude [13], [23]. Furthermore, unless one suitably selects the value of the constant, the spurious solutions also appear in the guided region [13], [15], [17]. On the other hand, Hano [14] has developed another vectorial finite-element method in terms of all three components of the electric and/or magnetic fields. In this procedure, spurious solutions do not appear, but needless zero eigenvalues are produced.

In this paper, a new finite-element method for the analysis of dielectric waveguide problems is developed in terms of the transverse magnetic-field component. In this approach, the relation $\nabla \cdot H = 0$ is satisfied and the spurious solutions are completely eliminated in the whole region of a propagation diagram. Furthermore, any artificial parameters such as the penalty coefficient that have been included in the three-component magnetic-field formulations [13], [15], [17] are not included, and the matrix is reduced to two-thirds the size of these formulations. To verify the accuracy of the method, numerical results for a rectangular metallic waveguide half filled with dielectric are presented and compared with exact [24] and earlier finite-element solutions [11]. Dielectric rectangular waveguides are also analyzed for both isotropic and anisotropic cases and the results obtained are compared with previously published results [25], [26].

II. Fundamental Equations

We consider a three-dimensional dielectric waveguide with an arbitrary cross section Ω in the xy-plane and assume that Γ, the boundary of the region Ω, consists partly of a perfect electric conductor and partly of a perfect magnetic conductor. With a time dependence of the form $\exp(j\omega t)$ being implied, from Maxwell's equations the following vectorial wave equation is derived:

$$\nabla \times ([K]^{-1} \nabla \times H) - k_0^2 H = 0 \tag{1}$$

where k_0 is the free-space wavenumber and $[K]$ is the relative permittivity tensor.

The divergence-free constraint $\nabla \cdot H = 0$ can be written

$$H_z = (1/j\beta)(\partial H_x/\partial x + \partial H_y/\partial y) \tag{2}$$

where β is the phase constant in the z-direction, assuming no losses.

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III. Finite-Element Formulation

Dividing the cross section Ω of the guide into a number of second-order triangular elements, the magnetic fields within each element are defined in terms of the magnetic fields at the corner and midside nodal points

$$H = [N]^T \{H\}_e \exp(-j\beta z) \tag{3}$$

where

$$[N] = \begin{bmatrix} \{N\} & \{0\} & \{0\} \\ \{0\} & \{N\} & \{0\} \\ \{0\} & \{0\} & j\{N\} \end{bmatrix} \tag{4}$$

and

$$\{H\}_e = \begin{bmatrix} \{H_x\}_e \\ \{H_y\}_e \\ \{H_z\}_e \end{bmatrix}. \tag{5}$$

Here, $\{N\}$ is the shape function vector, $\{0\}$ is a null vector, T, $\{\cdot\}$, and $\{\cdot\}^T$ denote a transpose, a column vector, and a row vector, respectively, and $\{H_x\}_e$, $\{H_y\}_e$, and $\{H_z\}_e$ are magnetic-field vectors corresponding to the nodal points within each element.

Application of the standard finite-element technique to (1) gives the following global matrix equation [9]–[16]:

$$[S]\{H\} - k_0^2[T]\{H\} = \{0\} \tag{6}$$

where the matrices $[S]$ and $[T]$ are related to the first and the second terms on the left-hand side of (1), respectively [9]–[16], and the nodal magnetic-field vector $\{H\}$ is forced to satisfy the boundary conditions on Γ [11], [16].

The solutions of (6) are known to include many spurious solutions [9]–[16] which do not satisfy the divergence relation (2).

Using the Galerkin procedure on (2), we obtain

$$\iint_e \{N\} H_z \, dx\, dy = (1/j\beta) \times \iint_e \{N\}(\partial H_x/\partial x + \partial H_y/\partial y)\, dx\, dy. \tag{7}$$

Equation (7) involves division by the phase constant β. Therefore, it may become unreliable when β is close to zero. To avoid such a difficulty, the following normalization is adopted here:

$$\iint_e \{N\} H_z \, d\bar{x}\, d\bar{y} = (1/j) \times \iint_e \{N\}(\partial H_x/\partial \bar{x} + \partial H_y/\partial \bar{y})\, d\bar{x}\, d\bar{y} \tag{8}$$

$$\bar{x} = \beta x, \quad \bar{y} = \beta y. \tag{9}$$

If β is exactly zero, another treatment is required since $\{H_t\}$ and $\{H_z\}$ are decoupled in (6). Substituting (3) into (8) and assembling the complete matrix for the region Ω by adding the contributions of all different elements, we obtain

$$[D_z]\{H_z\} = [D_t]\{H_t\} \tag{10}$$

where

$$[D_z] = \sum_e \iint_e \{N\}\{N\}^T d\bar{x}\, d\bar{y} \tag{11}$$

$$[D_t] = -\sum_e \iint_e [\{N\}\partial\{N\}^T/\partial\bar{x} \quad \{N\}\partial\{N\}^T/\partial\bar{y}]\, d\bar{x}\, d\bar{y} \tag{12}$$

$$\{H_t\} = \begin{bmatrix} \{H_x\} \\ \{H_y\} \end{bmatrix}. \tag{13}$$

Here, the components of vectors $\{H_x\}$, $\{H_y\}$, and $\{H_z\}$ are the values of H_x, H_y, and H_z at nodal points in Ω, respectively.

Using (10), the nodal magnetic-field vector $\{H\}$ can be expressed as

$$\{H\} = [D]\{H_t\} \tag{14}$$

where

$$[D] = \begin{bmatrix} [U] \\ [D_z]^{-1}[D_t] \end{bmatrix}. \tag{15}$$

Here, $[U]$ is a unit matrix.

Substituting (14) into (6) and multiplying (6) by $[D]^T$ from the left, we obtain the following final matrix equations with the transverse magnetic-field component $\{H_t\}$:

$$[\tilde{S}_{tt}]\{H_t\} - (k_0/\beta)^2[\tilde{T}_{tt}]\{H_t\} = \{0\} \tag{16}$$

where

$$[\tilde{S}_{tt}] = [D]^T[S][D] \tag{17}$$

$$[\tilde{T}_{tt}] = [D]^T[T][D]. \tag{18}$$

Equation (16) is an ordinary matrix eigenvalue problem whose eigenvalue and eigenvector are $(k_0/\beta)^2$ and $\{H_t\}$, respectively; therefore, one can easily manage with the help of a computer. Note that in (16) the divergence relation $\nabla \cdot H = 0$ is implicitly included and the matrix size solved is reduced to two-thirds that of the penalty function approach [11]–[13], [15]–[17].

IV. Numerical Examples

In this section, we present the computed results obtained by (16). In numerical computations, double precision is adopted to avoid roundoff errors. The matrix size solved is $2N_P$ and Householder's method is used as an eigenvalue solution method, where N_P is the number of nodal points.

A. Dielectric-Loaded Rectangular Waveguide

Fig. 1 shows the relative error e of the computed k_0 for the LSM_{11} mode in a rectangular metallic waveguide half filled with dielectric of relative permittivity 1.5. The relative error e is given by

$$e = \{(k_0 - \bar{k}_0)/\bar{k}_0\} \times 100[\%] \tag{19}$$

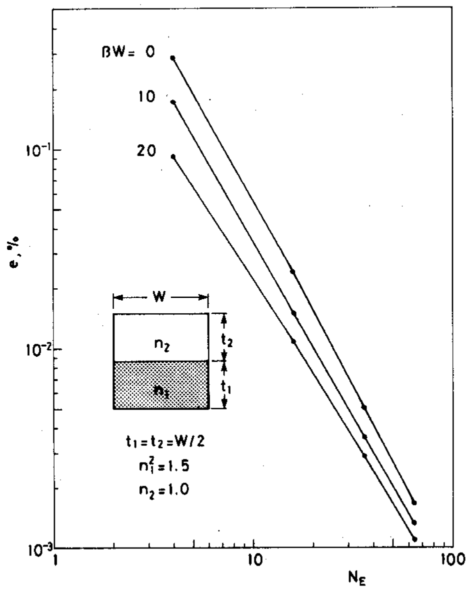

Fig. 1. Convergence of eigenvalues for a dielectric-loaded rectangular metallic waveguide.

TABLE I
COMPARISON WITH THE PENALTY FUNCTION METHOD

	k_0W	
No.	Present method	Penalty function method (p = 1)
1	8.8093	8.8095
2	9.3896	9.3882
3	10.2752	10.2765
4	11.1038	10.4819 (S)
5	11.2677	10.9431 (S)
6	11.4501	11.1058
7	11.9882	11.2736
8	12.6686	11.4644
9	12.8092	12.0014
10	12.9575	12.2266 (S)

S = spurious solutions. $\beta W = 10$.

where k_0 and $\bar{k}_0$ are the computed and exact values [24], respectively.

For the LSM_{11} mode, the ranges $0 \leqslant \beta W \leqslant 5.42$ and $5.42 \leqslant \beta W$ correspond to the fast-wave region ($0 \leqslant \beta/k_0 \leqslant 1$) and the slow-wave region ($1 \leqslant \beta/k_0$), respectively. It is readily seen from Fig. 1 that the relative error e monotonically decreases as the number of elements N_E increases. It is also found that for both the fast- and the slow-wave regions, the solution is always an upper bound. We confirm numerically that spurious solutions do not appear in either region.

Table I exhibits a list of the first ten eigenvalues obtained by the present method and compares them with those obtained by the penalty function method [11]. Here, $\beta W = 10$, p is the penalty coefficient [13], [15], and the plane of symmetry is assumed to be a perfect magnetic conductor. Four divisions, $(N_E, N_P) = (4,15)$, $(16,45)$, $(36,91)$, and $(64,153)$, are chosen in the numerical computations, and storage requirements are 6, 500, 2000, and 5700 kB, respectively. It is found from Table I that in the present method spurious solutions do not appear in either the fast- or the slow-wave region, while they do appear in the fast-wave region in the penalty function method. Also, it is interesting to note that the present method is better in accuracy than the penalty function method except for the second eigenvalue.

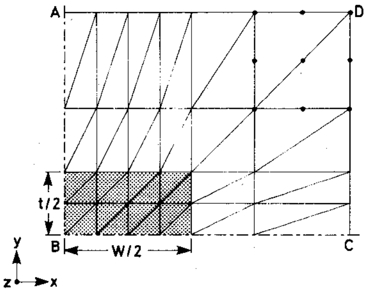

Fig. 2. Element division profile for a dielectric rectangular waveguide and second-order triangular elements.

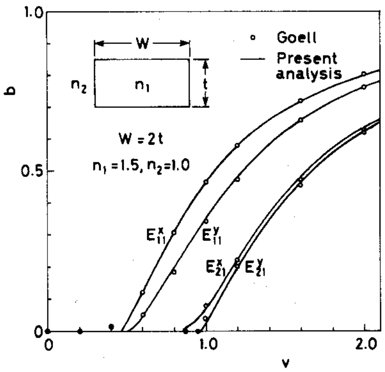

Fig. 3. Dispersion characteristics of an isotropic dielectric rectangular waveguide.

B. Dielectric Rectangular Waveguide

We subdivide only one-quarter of the cross section of a dielectric rectangular waveguide into second-order triangular elements ($N_E = 48$, $N_P = 117$) as shown in Fig. 2, where W and t are the width and the thickness of a rectangular core, respectively, boundaries CD and DA are assumed to be perfect electric conductors, and the conditions on boundaries AB and BC are suitably imposed depending on the kind of modes [11], [16]. The storage requirement is 3300 kB in the above division.

Fig. 3 shows the dispersion characteristics for the E_{pq}^x and E_{pq}^y modes [25] of an isotropic dielectric rectangular waveguide surrounded by a medium with a refractive index 1.0; the refractive index of the core is 1.5, $v = k_0 t\sqrt{n_1^2 - n_2^2}/\pi$, and $b = \{(\beta/k_0)^2 - n_2^2\}/(n_1^2 - n_2^2)$. Our results agree well with the results of the collocation method [25]. Goell's solution in Fig. 3 corresponds to the open structure and, consequently, shows no cutoff frequency, in contrast to the finite-element solution within a conducting box. The spurious solutions do not appear in the present analysis. In addition, it has already been confirmed that many spurious solutions are included in the solutions of (6) [11]–[13], [15]–[17].

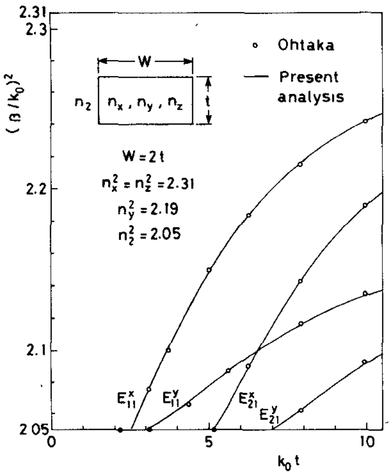

Fig. 4. Dispersion characteristics of an anisotropic dielectric rectangular waveguide.

Fig. 4 shows the dispersion characteristics for the E_{pq}^x and E_{pq}^y modes of an anisotropic dielectric rectangular waveguide surrounded by an isotropic medium with a refractive index of $\sqrt{2.05}$; the ordinary and extraordinary refractive indexes of the core are $\sqrt{2.31}$ and $\sqrt{2.19}$, respectively. Our results agree well with the results of the variational method [26]. Also, in this anisotropic case, the spurious solutions do not appear at all.

V. Conclusions

An improved finite-element method for the analysis of dielectric waveguiding problems has been formulated using the transverse magnetic-field component. In this approach, the divergence relation $\nabla \cdot H = 0$ is satisfied and the spurious solutions are perfectly eliminated in the entire region of a propagation diagram. Furthermore, any artificial parameters that have been included in the three-component magnetic- and/or electric-field formulations are not included, and the matrix size solved is reduced to two-thirds the size of these formulations. Although we have used the magnetic field here, one can also use the electric field by imposing the appropriate conditions [17], [19] on the boundaries of neighboring elements.

Unlike the previous formulations using transverse field components developed for use in the method of moments [27] and the finite-difference method [28], this approach is readily applicable to anisotropic waveguides having a permittivity tensor with nonzero off-diagonal elements and to waveguides containing lossy and/or active materials. This is because the permittivity tensor $[K]$ considered here is arbitrary except that it is assumed there are no losses and our formulation is based on a Galerkin procedure. Application of this technique to these other cases is to be made in the near future.

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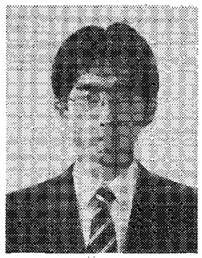

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